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The torsion problem for a circular cylinder with radial edge cracks

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SUMMARY
A finite Mellin transform technique reduces the torsion problem for a circular cylinder with radial edge cracks to that of solving some integral equations. Expressions are found for the stress intensity factors and crack formation energy. Three particular cases are considered in detail and numerical results given.

1. Introduction

Consider a cylinder $0 \leqq r \leqq b, 0 \leqq \theta \leqq 2\pi, 0 \leqq z \leqq L$ containing an array of edge cracks which are defined by the relations $0 < bc_i \leqq r \leqq b, \theta = \beta_i, 0 \leqq z \leqq L; i = 1, 2, 3, \ldots, n$ (Fig. 1). Let the length of the crack on the radius $\theta = \beta_i$ be denoted by $a_i = b(1 - c_i)$ and let the end $z = 0$ be fixed in the $r\theta$-plane. Then the problem we deal with is that of determining the stress intensity factors and the crack formation energy when the end $z = L$ is acted upon by a couple whose moment **T** lies along the z-aixs.

2. Reduction of the problem to integral equations

Let $\Omega_0 = \{(r, \theta): 0 \leqq r < b, 0 \leqq \theta < 2\pi\}$, $\Omega_i = \{(r, \beta_i): bc_i \leqq r \leqq b\}$, $i = 1,2,3,\ldots, n$ and $\Omega = \Omega_0 - \cup_{i=1}^{n} \Omega_i$. We assume that the couple acting on the end $z = L$ produces in the cylinder a twist of α per unit length and that the displacement field thus set up is of the form

$$u = 0, \quad v = \alpha rz, \quad w = \alpha\phi(r, \theta) \tag{2.1}$$

where u, v and w are the displacements in the r, θ and z-directions respectively. It is known [1] that the stresses corresponding to such a displacement field are given by the relations

$$\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{zz} = \sigma_{r\theta} = 0\,,$$

$$\sigma_{rz} = \mu\alpha \frac{\partial \phi}{\partial r} \quad \text{and} \quad \sigma_{\theta z} = \frac{\mu\alpha}{r} \frac{\partial \phi}{\partial \theta} + r^2\,, \tag{2.2}$$

where μ is the shear modulus, $\phi(r, \theta)$ is a solution of the partial differential equation

$$\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r} \frac{\partial \phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \phi}{\partial \theta^2} = 0\,, \tag{2.3}$$

$$\alpha = T/D\,, \tag{2.4}$$

and D (the torsional rigidity) is given by

$$D = \mu \iint_{\Omega} \left[\frac{\partial \phi}{\partial \theta} + r^2 \right] r dr d\theta\,. \tag{2.5}$$

It follows that the problem reduces to that of finding a function $\phi(r, \theta)$ which satisfies Eqn. (2.3) in the region Ω and is such that

$$(1)\ \frac{\partial \phi}{\partial r}(b, \theta) = 0, \qquad 0 \leqq \theta < 2\pi\,,$$

and

$$(2)\ \frac{\partial\phi}{\partial\theta}(r,\beta_i) = -r^2, \qquad bc_i \leqq r \leqq b\,.$$

Let

$$\phi(r,\theta) = \sum_{i=1}^{n} M_b^{-1}\left[\frac{A_i(s)}{s}\cdot\frac{\sin(\theta-\beta_i-\pi)s}{\sin\pi s}\,,\ s\to r\right], \tag{2.6}$$

where

$$A_i(s) = \int_{bc_i}^{b} \frac{P_i(t)}{\sqrt{[(b-t)(t-bc_i)]}}(b^{2s}t^{-s}-t^s)dt\,, \tag{2.7}$$

$i=1,2,3,\ldots,n$ $|\mathrm{Re}(s)|<1$ and M_b^{-1} is the inverse of the finite Mellin transform

$$M_b[f(r);s] = \int_0^b f(r)[r^{s-1}+b^{2s}r^{-s-1}]\mathrm{d}r\,. \tag{2.8}$$

$\phi(r,\theta)$ is a solution of Eqn. (2.3) in Ω [2], it satisfies condition (1) and is such that

$$\phi(r,\beta_i^+)-\phi(r,\beta_i^-) = \begin{cases} -2\displaystyle\int_{bc_i}^{r}\frac{P_i(t)}{\sqrt{[(b-t)(t-bc_i)]}}dt, & bc_i\leqq r\leqq b\,, \\ 0, & 0\leqq r\leqq bc_i\,, \end{cases} \tag{2.9}$$

Furthermore,

$$\begin{aligned}\frac{\partial\phi}{\partial\theta}(r,\theta) &= \sum_{i=1}^{n} M_b^{-1}\left[A_i(s)\,\frac{\cos(\theta-\beta_i-\pi)s}{\sin\pi s}\ ;\ s\to r\right] \\ &= \sum_{i=1}^{n}\int_{bc_i}^{b}\frac{P_i(t)}{\sqrt{[(b-t)(t-bc_i)]}}M_b^{-1}\left[\frac{(b^{2s}t^{-s}-t^s)\cos(\theta-\beta_i-\pi)s}{\sin\pi s}\ ;\ r\right]dt\,,\end{aligned}$$

and hence (2) will be satisfied if

$$\frac{1}{\pi}\sum_{j=1}^{n}\int_{bc_j}^{b}\frac{P_j(t)}{\sqrt{[(b-t)(t-bc_j)]}}K(r,\beta_i-\beta_j,t)dt = -r^2, \qquad bc_i<r<b\,, \tag{2.10}$$

where

$$\begin{aligned}K(r,\theta,t) &= \pi M_b^{-1}\left[\frac{(b^{2s}t^{-s}-t^s)\cos(\theta-\pi)s}{\sin\pi s}\ ;\ r\right] \\ &= \frac{1}{2}\left\{\frac{r^2-t^2}{r^2-2rt\cos\theta+t^2}+\frac{b^4-r^2t^2}{r^2t^2-2b^2rt\cos\theta+b^4}\right\}.\end{aligned} \tag{2.11}$$

Also, since

$$\frac{\partial\phi}{\partial r}(b,\beta_i)=0\,,$$

we have

$$P_j(b)=0\,, \tag{2.12}$$

$j=1,2,3,\ldots,n$. The problem is thus reduced to that of solving the integral equations (2.10) subject to the subsidiary conditions (2.12).

3. The torsional rigidity

By Eqns. (2.5) and (2.9) we see that the torsional rigidity D is given by the formula

$$D = 2\mu\sum_{i=1}^{n}\int_{bc_i}^{b} r\int_{bc_i}^{r}\frac{P_i(t)dt\,dr}{\sqrt{[(b-t)(t-bc_i)]}}+\frac{\pi}{2}\mu b^4\,. \tag{3.1}$$

Therefore, on changing the order of integration,

$$D = \frac{\mu b^4}{\pi} \left[\frac{\pi^2}{2} - I_n \right], \tag{3.2}$$

where

$$I_n = \frac{\pi}{b^4} \sum_{i=1}^{n} \int_{bc_i}^{b} \frac{P_i(t)}{\sqrt{[(b-t)(t-bc_i)]}} (t^2 - b^2) dt . \tag{3.3}$$

4. The stress intensity factors and crack formation energy

The stress intensity factor of the tip at the point (bc_i, β_i) is defined by the equation

$$K^{(i)} = \frac{\mu}{2} \underset{r \to bc_i+}{\text{limit}} \sqrt{[2(r-bc_i)]} \frac{\partial}{\partial r} [w(r, \beta_i+) - w(r, \beta_i-)] \tag{4.1}$$

and therefore, by Eqns. (2.1) and (2.9), we have

$$K^{(i)} = -\frac{\sqrt{2}}{\sqrt{[b(1-c_i)]}} \cdot \frac{\mu T P_i(bc_i)}{D} . \tag{4.2}$$

On substituting from Eqn. (3.2) into this equation and simplifying we see that

$$\frac{K^{(i)} a_i^{\frac{5}{2}}}{T} = -\frac{\pi\sqrt{2}}{\left[\frac{\pi^2}{2} - I_n \right]} \left(\frac{1-c_i}{b} \right)^2 P_i(bc_i) , \tag{4.3}$$

where $a_i = b(1-c_i)$ is the length of the crack on the radius $\theta = \beta_i$.

Similarly, the crack formation energy W is defined by the equation

$$W = \frac{1}{2} \sum_{i=1}^{n} \int_{bc_i}^{b} \sigma_{\theta z}^{(i)}(r) [w(r, \beta_i+) - w(r, \beta_i-)] dr , \tag{4.4}$$

where $\sigma_{\theta z}^{(i)}(r)$ is the shear stress on the line $\theta = \beta_i$ in the absence of the crack. Therefore, by Eqns. (2.1) and (2.9), we have

$$W = \frac{\mu \alpha^2 b^4}{2\pi} \cdot I_n , \tag{4.5}$$

where I_n is given by Eqn. (3.3). But $\alpha = T/D$ and hence on making use of Eqn. (3.2) we find that

$$W = \frac{\pi T^2 I_n}{2\mu b^4 [\pi^2/2 - I_n]^2} . \tag{4.6}$$

Let $W_0 = \sum_{i=1}^{n} W_i$, where

$$W_i = \frac{T^2}{\pi \mu b^4} (1-c_i)^2 \tag{4.7}$$

is the formation energy of an edge crack of length $b(1-c_i)$ in a semi-infinite sheet which is subject to a constant shear load $2T/\pi b^3$ at infinity, then clearly by Eqn. (4.7) we have

$$\frac{W}{W_0} = \frac{\pi^2 I_n}{2[\pi^2/2 - I_n]^2 \sum_{i=1}^{n} (1-c_i)^2} . \tag{4.8}$$

5. Special cases

Case (a)

The first special case we consider is that in which the cylinder contains two cracks of length

a defined by the relations $bc \leqq r \leqq b, \theta = \beta$ or $-\beta, 0 \leqq z \leqq L$ and $a = b(1-c)$. In this case $P_1(r) = P_2(r)$ and if we let $\tau = t/b$, $\rho = r/b$ and put $Q(\tau) = b^{-2} P_1(b\tau)$, Eqns. (2.10) and (2.12) become

$$\frac{1}{\pi} \int_c^1 \frac{Q(\tau)}{\sqrt{[(1-\tau)(\tau-c)]}} K_1(\rho, \beta, \tau) d\tau = -\rho^2, \qquad c < \rho < 1, \tag{5.1}$$

and

$$Q(1) = 0, \tag{5.2}$$

where

$$K_1(\rho, \beta, \tau) = \frac{\tau}{\rho - \tau} + \frac{1}{1 - \rho\tau} + \frac{\rho^2 - \tau^2}{2(\rho^2 - 2\rho\tau\cos 2\beta + \tau^2)} + \frac{1 - \rho^2\tau^2}{2(\rho^2\tau^2 - 2\rho\tau\cos 2\beta + 1)}. \tag{5.3}$$

Following Erdogan and Gupta [3] we approximate Eqns. (5.1) and (5.2) by the linear algebraic system

$$\frac{1}{m} \sum_{j=1}^{m} Q(\tau_j) K_1(\rho_i, \beta, \tau_j) = -\rho_i^2, \qquad i = 1, 2, 3, \ldots, m-1, \tag{5.3}$$

$$\frac{1}{m} \sum_{j=1}^{m} (-)^j \left(\frac{1+x_j}{1-x_j}\right)^{\frac{1}{2}} Q(\tau_j) = 0,$$

where $x_j = \cos[(2j-1)\pi/2m]$, $\tau_j = \frac{1}{2}(1-c)x_j + \frac{1}{2}(1+c)$, $j = 1, 2, 3, 4, \ldots, m$ and $\rho_i = \frac{1}{2}(1-c)\cos(\pi i/m) + \frac{1}{2}(1+c)$, $i = 1, 2, 3, 4, \ldots, m-1$. On solving these equations for the unknowns $Q(\tau_j)$ we calculate I_2, $Ka^{\frac{5}{2}} T^{-1}$ and W/W_0 from the formulae

$$I_2 = \frac{2\pi^2}{m} \sum_{j=1}^{m} (\tau_j^2 - 1) Q(\tau_j), \tag{5.4}$$

$$\frac{Ka^{\frac{5}{2}}}{T} = -\frac{2\pi a^2 \sqrt{2}}{m[\pi^2 - 2I_2]} \sum_{j=1}^{m} (-)^{j+m} \left(\frac{1-x_j}{1+x_j}\right)^{\frac{1}{2}} Q(\tau_j), \tag{5.5}$$

and

$$\frac{W}{W_0} = \frac{\pi^2 I_2}{[\pi^2 - 2I_2]^2 (1-c)^2}. \tag{5.6}$$

The results of such a calculation are shown graphically in Figures 2 and 3 which show respectively the variation of $Ka^{\frac{5}{2}} T^{-1}$ and W/W_0 with a/b for several values of β.

Case (b)

Next we consider a symmetrical array of cracks defined by the equations $bc \leqq r \leqq b, \theta = 2k\pi/n$, $k = 0, 1, 2, \ldots, n-1$, $0 \leqq z \leqq L$. In this case $a_i = b(1-c) = a$, $P_i(r) = P(r)$, $i = 1, 2, \ldots, n$ and we can write Eqns. (2.10) and (2.12) in the form

$$\frac{1}{\pi} \int_c^1 \frac{Q(\tau)}{\sqrt{[(1-\tau)(\tau-c)]}} K_2(\rho, n, \tau) d\tau = -\rho^2, \quad c < \rho < 1, \tag{5.7}$$

$$Q(1) = 0,$$

where $\tau = t/b$, $\rho = r/b$, $Q(\tau) = b^{-2} P(b\tau)$ and

$$K_2(\rho, n, \tau) = \frac{\tau}{\rho - \tau} + \frac{1}{1 - \rho\tau}$$

$$+ \frac{1}{2} \sum_{j=2}^{n} \left[\frac{\rho^2 - \tau^2}{\rho^2 - 2\rho\tau\cos[2(j-1)\pi/n] + \tau^2} + \frac{1 - \rho^2\tau^2}{1 - 2\rho\tau\cos[2(j-1)\pi/n] + \rho^2\tau^2}\right]. \tag{5.8}$$

As in the previous case we solve Eqns (5.7) by the method of Erdogan and Gupta and then use formulae of the type (5.4)–(5.6) to find $Ka^{\frac{5}{2}} T^{-1}$ and W/W_0. The results for this case are shown graphically in Figs. 4 and 5.

Case (c)

The last case we consider is that in which the cylinder contains two unequal cracks on the same diameter. If these are defined by the relations $bc_i \leqq r \leqq b$, $\theta = (i-1)\pi$, $0 \leqq z \leqq L$, $a_i = b(1-c_i)$, $i=1, 2$ and if $\tau = t/b$, $\rho = r/b$, $Q_1(\tau) = b^{-2} P_1(b\tau)$ and $Q_2(\tau) = b^{-2} P_2(-b\tau)$ then Eqns. (2.10) and (2.12) take the form

$$\frac{1}{\pi}\int_{-1}^{-c_2} \frac{Q_2(\tau) K_3(\rho, \tau) d\tau}{\sqrt{[(-c_2-\tau)(\tau+1)]}} + \frac{1}{\pi}\int_{c_1}^{1} \frac{Q_1(\tau) K_2(\rho, \tau) d\tau}{\sqrt{[(1-\tau)(\tau-c_1)]}} = -\rho^2 ,$$

$$(-1 < \rho < -c_2) \cup (c_1 < \rho < 1) , \quad (5.9)$$

$$Q_2(-1) = Q_1(1) = 0 ,$$

where

$$K_3(\rho, \tau) = \frac{\tau}{\rho - \tau} + \frac{1}{1 - \rho\tau} . \quad (5.10)$$

Again the problem is solved by the method of Erdogan and Gupta which yields the results shown in Figs. 6 and 7.

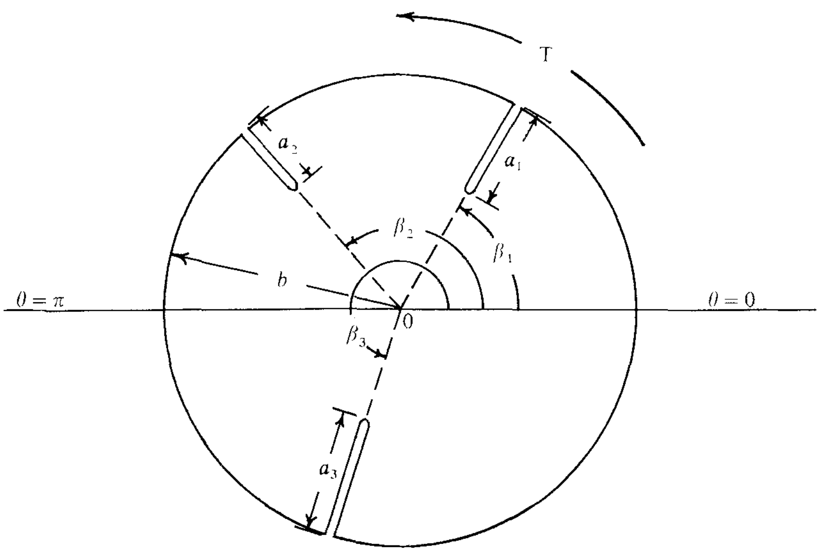

Figure 1.

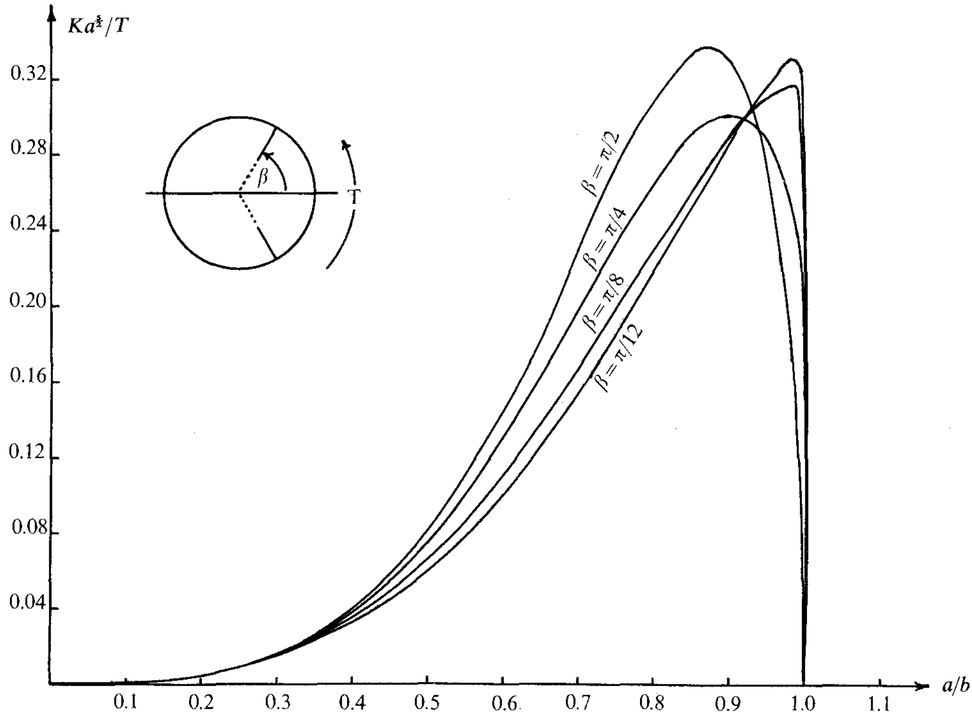

Figure 2. The variation of $Ka^{\frac{1}{2}}/T$ with a/b for several values of β.

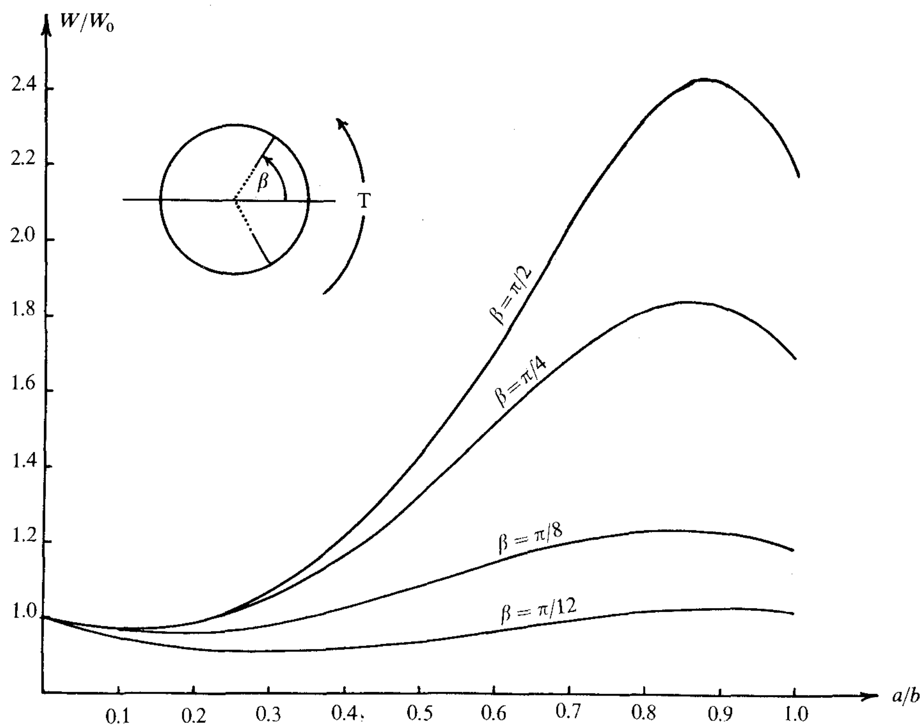

Figure 3. The variation of W/W_0 with a/b for several values of β.

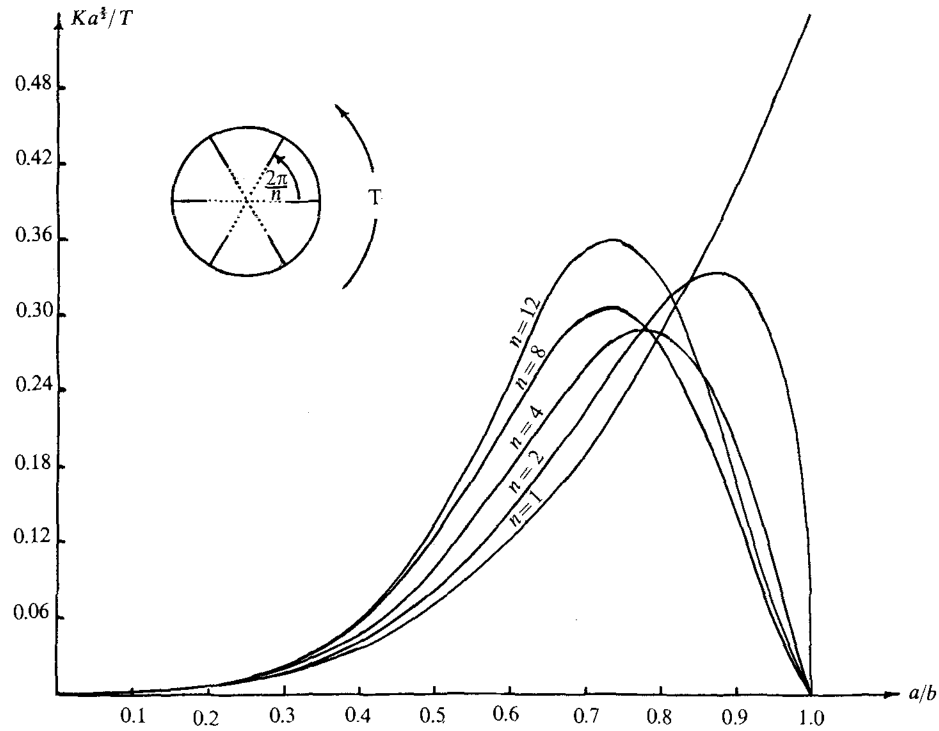

Figure 4. The variation of $Ka^{\frac{1}{2}}/T$ with a/b for several values of n.

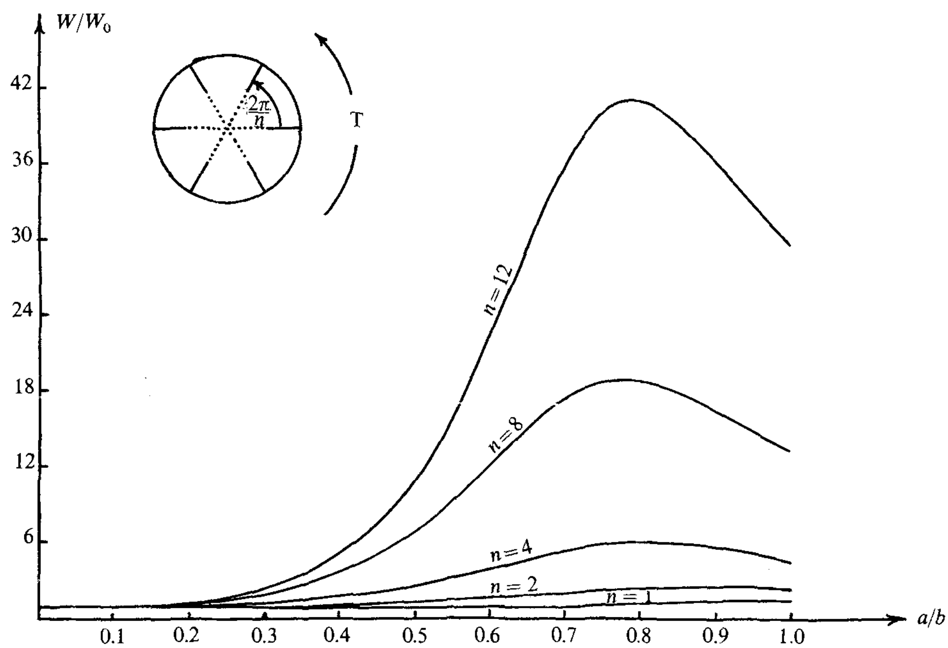

Figure 5. The variation of W/W_0 with a/b for several values of n.

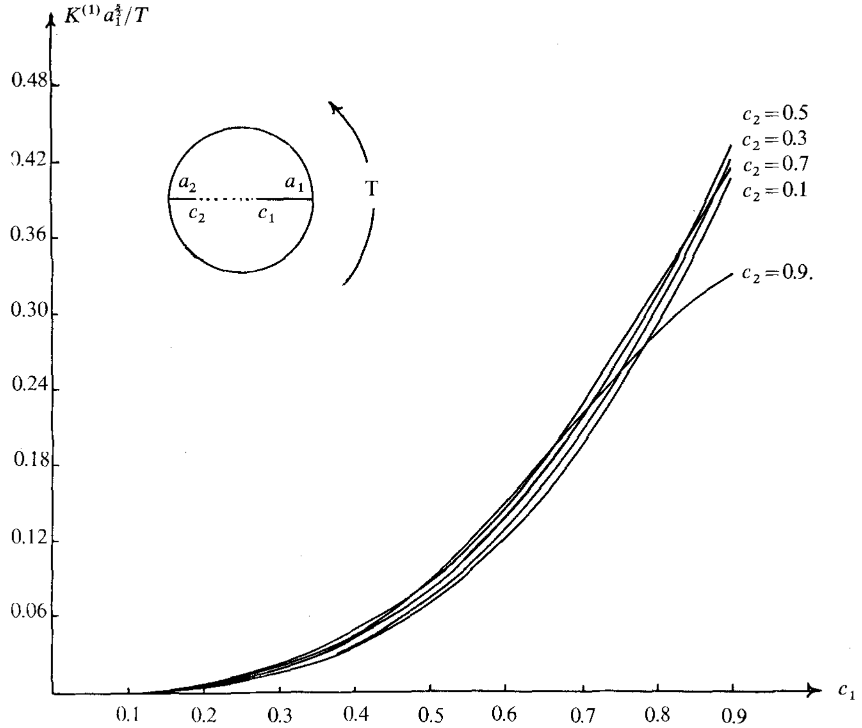

Figure 6. The variation of $K^{(1)}a_1^{\frac{3}{2}}/T$ with c_1 for several values of c_2.

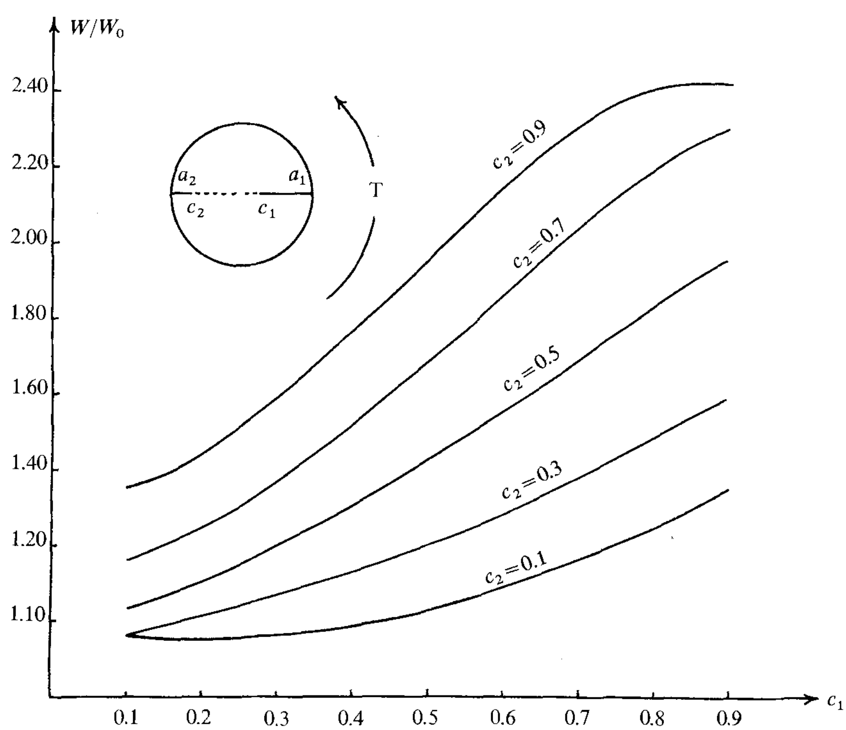

Figure 7. The variation of W/W_0 with c_1 for several values of c_2.

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